Lecture 1

- Review of 1-dim integration
- Double integral over a rectangle

1.1. Review of 1-dim Integration

Let f be a function on $[a,b]$.

A partition P is a set of points

$$P = \{x_j : j=0,1,\dots,n,\ x_0 = a, x_n = b,\ x_0 < x_1 < x_2 < \dots < x_n\}.$$

A tag is a collection of points $z_1,\dots,z_n$, $z_j \in [x_{j-1}, x_j]$.

$\dot{P} = (P, \{z_j\})$ a tagged partition.

Given a tagged partition, its Riemann sum is

$$S(f,\dot{P}) = \sum_{j=1}^{n} f(z_j)\Delta x_j \qquad \text{where } \Delta x_j = x_j - x_{j-1} > 0.$$

When $f \geq 0$ on $[a,b]$, $S(f,\dot{P})$ represents approximate area of the region bounded by $y=f(x)$, x-axis, $x=a$ and $x=b$.

<u>Definition 1</u> f is integrable over $[a,b]$ if $\exists\, I \in \mathbb{R}$ such that $\forall \varepsilon > 0$, $\exists\, \delta$ so that

$$\left| S(f,\dot{P}) - I \right| < \varepsilon, \text{ for all } P,\ \|P\| < \delta.$$

Equivalently,

$$-\varepsilon < S(f,\dot{P}) - I < \varepsilon.$$

Here the norm of P

$$\|P\| = \max\{\Delta x_1, \Delta x_2, \ldots \Delta x_n\}.$$

e.g 1

$$g(x) = \begin{cases} 1, & x \text{ irrational} \\ 0, & x \text{ rational}. \end{cases}$$

We claim that g is not integrable over any $[a,b]$.

Let P be any partition. Choose tags $z_j \in [x_{j-1}, x_j]$ to be rational numbers. For this tagged partition,

$$S(g,\dot{P}) = \sum_{j=1}^{n} g(z_j)\Delta x_j$$
$$= \sum_{j=1}^{n} 0 \, \Delta x_j$$
$$= 0.$$

On the other hand, choose tags $w_j \in [x_{j-1}, x_j]$ to be irrational. For this $\mathring{P}$,

$$S(g,\mathring{P}) = \sum_{j=1}^{n} g(w_j)\Delta x_j$$
$$= \sum_{j=1}^{n} 1 \, \Delta x_j$$
$$= b-a.$$

As $0 \neq b-a$, $S(g,\dot{P})$ and $S(g,\mathring{P})$ can't approach the same number I. $\square$

Luckily, we have

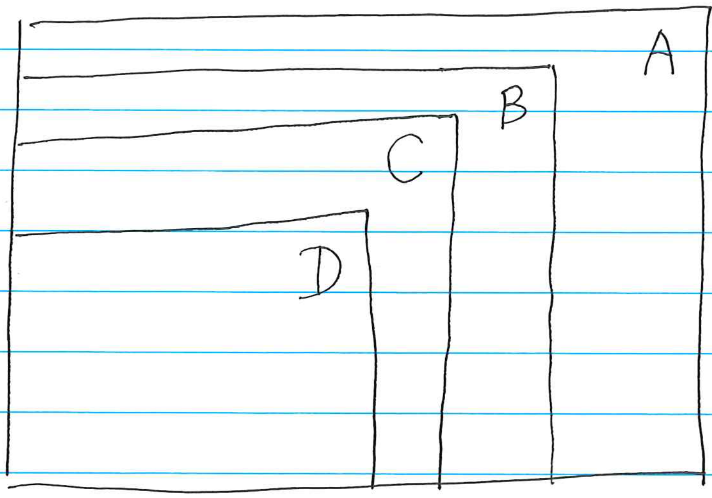
Theorem 1 f continuous on $[a,b] \Rightarrow f$ integrable on $[a,b]$.

(In fact, f is still integrable if it has finitely many discontinuous pts only.)

Theorem 2 f integrable on $[a,b] \Rightarrow f$ is bounded on $[a,b]$.

(f is bounded on $[a,b]$ if $\exists M$ s.t. $|f(x)| \leq M$, $\forall x \in [a,b]$.)

e.g. 2. $h(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ 0 & x = 0 \end{cases}$ is not bounded on $[0,1]$.



$D \subsetneq C \subsetneq B \subsetneq A$

D differentiable fcns

C continuous fcns

B integrable fcns

A bounded fcns on $[a,b]$

Notation for the integral :

$$\int_a^b f \ , \ \int_{[a,b]} f \ , \ \int_a^b f(x)\,dx \ , \ \int_a^b f(t)\,dt, \text{ etc.}$$

Thanks to Newton we don't need to sum up $S(f, \mathring{P})$ to obtain the integral. The recipe depends on the anti-derivative of f (also called indefinite integral, primitive function, etc).

F is called the antiderivative of f if

$$F'(x) = f(x).$$

Fundamental Theorem of Calculus : Let f be continuous on $[a,b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

1.2 Double Integral

Let f be a function on $[a,b]\times[c,d]$.

A partition P on $[a,b]\times[c,d]$ is

$$P=\{x_j, y_k : a=x_0<x_1<\cdots<x_n=b,\ c=y_0<y_1<\cdots<y_m=d\}$$

A tag is a collection of points $p_{jk}\in[x_{j-1},x_j]\times[y_{k-1},y_k]$

$\dot{P}=(P,\{p_{jk}\})$ a tagged partition.

Given $\dot{P}$, its Riemann sum is

$$S(f,\dot{P})=\sum_{j,k} f(p_{jk})\,\Delta x_j\,\Delta y_k$$

$$\overset{\text{or}}{=}\sum_{j=1}^{n}\sum_{k=1}^{m} f(p_{jk})\,\Delta y_k\,\Delta x_j$$

$$\overset{\text{or}}{=}\sum_{k=1}^{m}\sum_{j=1}^{n} f(p_{jk})\,\Delta x_j\,\Delta y_k$$

When $f\geq 0$, $S(f,\dot{P})$ represents an approximate volume of the region over $[a,b]\times[c,d]$, bounded by $z=0$ and $z=f(x,y)$

<u>Definition 1'</u> f is integrable over $[a,b]\times[c,d]$ if $\exists\, I$ s.t. $\forall\, \varepsilon>0$, $\exists\,\delta$ so that

$$|S(f,\dot{P})-I|<\varepsilon,\quad \text{for all } P,\ \|P\|<\delta$$

and any tag on P.

Here the norm of P, $\|P\|$, is

$$\|P\| = \max\{\Delta x_1, \ldots, \Delta x_n, \Delta y_1, \ldots, \Delta y_m\}.$$

When exists, I is the volume of the region bounded by $z=0$, $z=f(x,y)$ and over $[a,b]\times[c,d]$.

Theorem 1' f continuous on $[a,b]\times[c,d]$

$\Longrightarrow f$ integrable on $[a,b]\times[c,d]$.

Theorem 2' f integrable on $[a,b]\times[c,d]$

$\Longrightarrow f$ is bounded on $[a,b]\times[c,d]$.

The situation is similar to the 1-dim case so far.

The evaluation of double integral depends on the following result.

Theorem 3 (Fubini's theorem) Let f be continuous on $[a,b]\times[c,d]$. Then

$$\iint\limits_{[a,b]\times[c,d]} f = \int_a^b \left(\int_c^d f(x,y)\,dy\right)dx$$

$$\overset{\text{or}}{=} \int_c^d \left(\int_a^b f(x,y)\,dx\right)dy.$$

Notation for I :

$$\iint_{[a,b]\times[c,d]} f \quad , \quad \iint_{[a,b]\times[c,d]} f(x,y)\,dx\,dy \quad , \quad \iint_{[a,b]\times[c,d]} f(x,y)\,dA \quad , \text{ etc}$$

e.g. 3. Find

$$\int_0^2 \int_0^1 x y^2 \,dx\,dy .$$

By Fubini's

$$\iint_{[0,1]\times[0,2]} x y^2 = \int_0^2 \left(\int_0^1 x y^2 \,dy \right) dx$$

$$= \int_0^2 \left. \frac{x y^3}{3} \right]_{y=0}^{y=1} dx$$

$$= \int_0^2 \frac{x}{3} \,dx$$

$$= \left. \frac{x^2}{6} \right|_0^2$$

$$= \frac{2}{3} .$$

Or,

$$\iint_{[0,1]\times[0,2]} x y^2 = \int_0^1 \left(\int_0^2 x y^2 \,dx \right) dy$$

$$= \int_0^1 \left. \frac{x^2}{2} y^2 \right|_{x=0}^{x=2} dy$$

$$= \int_0^1 2y^2\,dy$$

$$= \frac{2}{3}y^3\Big|_0^1$$

$$= \frac{2}{3}, \text{ the same result.}$$

e.g. 4 Find $\displaystyle\iint_{[0,1]\times[0,\pi]} x \sin(xy)\,dA$.

$$\iint_{[0,1]\times[0,\pi]} x\sin(xy)\,dA = \int_0^{\pi}\left(\int_0^1 x\sin(xy)\,dx\right)dy$$

$$= \int_0^{\pi}\left(-\frac{\cos y}{y} + \frac{\sin y}{y^2}\right)dy$$

(after $\int$ by parts)

$= ?$ hard to proceed further!

$$\iint_{[0,1]\times[0,\pi]} x\sin(xy)\,dA = \int_0^1\left(\int_0^{\pi} x\sin(xy)\,dy\right)dx$$

$$= \int_0^1 x \times \frac{-\cos xy}{x}\Big|_{y=0}^{y=\pi}\,dx$$

$$= \int_0^1 (-\cos \pi x + 1)\,dx$$

$$= \left(-\frac{\sin \pi x}{\pi} + x\right)\Big|_{x=0}^{x=1}$$

$= 1$, done!